2015 IEEE International Conference on Computational Intelligence & Communication Technology

Deterministic Equivalent for the Analysis of MIMO Multiple Access Channel Capacity

Zhijiao Zhang, Dazhuan Xu, Hao Luo, Xiaofei Zhang
College of Electronic and Information Engineering
Nanjing University of Aeronautics and Astronautics
Nanjing, CHINA
zhangzhijiao1104@126.com

***Abstract*—In this paper we consider the multiple-input multiple-output (MIMO) multiple access channel with antenna correlation at the transmitter and receiver. The propagation channels between transmitters and receivers are modelled according to the Kronecker model. We derive the specific deterministic equivalent of ergodic capacity and the optimal precoding matrices based on large dimensional random matrix theory (RMT). Simulations confirm the theoretical claims and also indicate that in different fading scenarios the results applied to a finite number of antennas give good approximations.**

Keywords—Deterministic equivalent; ergodic capacity; multiple access channel; multiuser MIMO

I. INTRODUCTION

Recent years wireless systems continue to seek after higher data rates because of limited frequency resources [1]. The use of multiple transmit and receive antennas first studied by Winters [2], Foschini [3], and Telatar [4] significantly increase channel capacity. This caused a great deal of attention and resulted in a lot of research into the theoretical and practical issues associated with multiple-input multiple-output (MIMO) wireless channels and extension of multiuser system. Much subsequent work has been done characterizing MIMO channel under more realistic channel model [1]. Channel correlation is one factor that should not be neglected. Thanks to the main contributions to the study of the effects of correlation to wireless communications [5], we can focus on the capacity of MIMO systems with different types of antenna correlation varying drastically as a function of the scattering environment, the distance between antennas, etc.

In the present article, we study the performance of MIMO multiple access channel (MIMO-MAC) where several users communicate with a multiple-antenna base station simultaneously. Goldsmith et al. provide an overview of the extensive recent results on the Shannon capacity of single-user and multiuser MIMO channels in [1]. The article clearly point out that in systems with moderate to high user mobility the system designer is inevitably faced with channels that change rapidly. Fading models where only the channel distribution is available to the transmitter are more applicable to such systems. So we continue the following research under such circumstances. In [6], Kiessling derive exact formulas of the ergodic capacity, thus unifying the capacity analysis of correlated Rayleigh fading point-to-point MIMO channels. It turns out that the ergodic capacity in all cases can be expressed in terms of a sum of determinants with elements that are a combination of polynomials, exponentials and the exponential integral. Without Monte-Carlo simulations, we can calculate mutual information of Rayleigh fading MIMO channels with arbitrary fading correlation at transmitter and receiver directly [6]. In [7], a deterministic equivalent of the mutual information of MIMO-MAC is derived utilizing large dimensional random matrix theory (RMT). As we all know, when a system is sufficiently large exact performance is too complex to analyse directly. Large dimensional RMT provides a powerful tool in dealing with large-scale MIMO systems [8]. We therefore conclude two theorems supporting our theoretical research on ergodic capacity according to [7]. Moreover, an approximation of the linear precoders that achieve the boundary of the rate region as well as an iterative water-filling algorithm to obtain these precoders are provided in [7]. However, the details of the algorithms are not given and just one circumstance is considered in simulations.

The aim of this paper is to calculate the capacity of MIMO-MAC in detail and obtain optimal precoding matrices to maximize capacity. In addition, we consider several types of fading distribution in simulation to prove the feasibility of proposed algorithms. We have shown that the deterministic equivalent provides a very good approximation for the capacity in different fading channel and takes much less time than Monte-Carlo simulations especially in large-scale MIMO system. Moreover, the water-filling algorithm increases the per-antenna rate compared to the uniform power allocation.

II. SYSTEM MODEL

Consider a wireless multiuser system in which the base station has N antennas and each of the K users has n_k antennas. The uplink of this system is a MIMO MAC and the downlink is a MIMO broadcast channel (BC). Use $\mathbf{H}_k$ to denote the uplink matrix of user k and $\mathbf{G}_k$ to denote the downlink matrix from the base station to user k.

In the MAC, let $\mathbf{s}_k \in C^{n_k}$ be the transmitted signal of user k. Let $\mathbf{y} \in C^N$ denote the received signal and

978-1-4799-6023-1/15 $31.00 © 2015 IEEE
DOI 10.1109/CICT.2015.42

$\mathbf{n} \sim \mathrm{N}\left(0, \sigma^2 \mathbf{I}_N\right)$ the additive white Gaussian noise. The received signal at base station is then given by

$$\mathbf{y} = \mathbf{H}_1 \mathbf{s}_1 + \ldots + \mathbf{H}_K \mathbf{s}_K + \mathbf{n} = \mathbf{H} \begin{bmatrix} \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_K \end{bmatrix} + \mathbf{n} \quad (1)$$

where $\mathbf{H} = [\mathbf{H}_1 \ \ldots \ \mathbf{H}_K]$.

In the BC, let $\mathbf{x} \in C^N$ denote the transmitted vector signal from base station. The noise at user k is represented by $\mathbf{w}_k \in C^{n_k}$. The received signal of user k is equal to

$$\mathbf{r}_k = \mathbf{G}_k \mathbf{x} + \mathbf{w}_k \quad (2)$$

The focus of this paper is on the uplink channel of multiuser MIMO. Let $\mathrm{E}[\mathbf{s}_k] = 0$ and $\mathrm{E}[\mathbf{s}_k \mathbf{s}_k^H] = \mathbf{P}_k$, with $1/n_k \, tr\mathbf{P}_k \le P_k$ where P_k is the total power of user k. We further assume that the channel $\mathbf{H}_k$ follows the widely used Kronecker model

$$\mathbf{H}_k = \mathbf{R}_k^{\frac{1}{2}} \mathbf{X}_k \mathbf{T}_k^{\frac{1}{2}} \quad (3)$$

where $\mathbf{X}_k \in C^{N \times n_k}$ is a matrix with i.i.d Gaussian entries of zero mean and variance $1/n_k$. $\mathbf{T}_k \in C^{n_k \times n_k}$ and $\mathbf{R}_k \in C^{N \times N}$ are Hermitian correlation matrices at transmitter k and receiver, respectively. Claim that the diagonal entries of $\mathbf{R}_k$ and $\mathbf{T}_k$ are equal to one, which leads to $tr\mathbf{R}_k = N$ and $tr\mathbf{T}_k = n_k$.

III. MIMO-MAC Capacity

In this section, we successively derive the algorithm to approximate the ergodic per-antenna rate for all deterministic precoders based on the above model, and then determine the precoders that maximize this approximated mutual information according to [7]. We also get an iterative power allocation algorithm to obtain the optimal precoders for reaching the boundary of MIMO-MAC rate region.

A. Deterministic Equivalent of Capacity

The per-antenna ergodic rate C_{MAC} of the MAC channel under respective precoders $\mathbf{P}_1, \ldots, \mathbf{P}_K$ for users $1, \ldots, K$ is given in [7]

$$C_{MAC} = \mathrm{E}\left[\frac{1}{N} \log\det\left(\mathbf{I}_N + \frac{1}{\sigma^2} \sum_{i \in S} \mathbf{H}_i \mathbf{P}_i \mathbf{H}_i^{\mathrm{H}}\right)\right], \forall S \in \{1, \ldots, K\} \quad (4)$$

Depending on the large dimension RMT, we have the deterministic equivalent of C_{MAC} [7]

$$\begin{aligned} C_{MAC}^0 = & \sum_{k=1}^{K} \frac{1}{N} \log\det\left(\mathbf{I}_{n_k} + c_k e_k\left(-\sigma^2\right) \mathbf{T}_k \mathbf{P}_k\right) \\ & + \frac{1}{N} \log\det\left(\mathbf{I}_N + \sum_{k=1}^{K} \delta_k\left(-\sigma^2\right) \mathbf{R}_k\right) \\ & - \sigma^2 \sum_{k=1}^{K} \delta_k\left(-\sigma^2\right) e_k\left(-\sigma^2\right) \end{aligned} \quad (5)$$

where

$$\begin{aligned} \delta_i^n\left(-\sigma^2\right) &= \frac{1}{n_i} tr \mathbf{T}_i^{\frac{1}{2}} \mathbf{P}_i \mathbf{T}_i^{\frac{1}{2}} \left(\sigma^2 \left[\mathbf{I}_N + c_i e_i^n\left(-\sigma^2\right) \mathbf{T}_i^{\frac{1}{2}} \mathbf{P}_i \mathbf{T}_i^{\frac{1}{2}}\right]\right)^{-1} \\ e_i^{n+1}\left(-\sigma^2\right) &= \frac{1}{N} tr \mathbf{R}_i \left(\sigma^2 \left[\mathbf{I}_N + \sum_{k=1}^{K} \delta_k^n\left(-\sigma^2\right) \mathbf{R}_k\right]\right)^{-1} \end{aligned} . \quad (6)$$

An iterative algorithm is proposed to calculate $e_i\left(-\sigma^2\right)$ and $\delta_i\left(-\sigma^2\right)$ according to [9] which puts forward a fixed-point algorithm depending on the convergence of $e_i\left(-\sigma^2\right)$ in Table I.

The per-antenna rate has been a key metric for performance analysis of a MAC [8]. To derive the deterministic equivalent in (5), we need to calculate e_j^{n+1} and δ_j^{n+1} in Table I first instead of $e_i\left(-\sigma^2\right)$ and $\delta_i\left(-\sigma^2\right)$.

B. Water-filing Algorithm

We have shown the deterministic equivalent for ergodic

TABLE I. FIXED-POINT ALGORITHM

Initialization: Define the iteration step $n \ge 0$. $e_j^0 = 1/\sigma^2$ for all $i \in \{1, \ldots, K\}$.
Iteration n:
for $i \in \{1, \ldots, K\}$ calculate
$\delta_i^n\left(-\sigma^2\right) = \frac{1}{n_i} tr \mathbf{T}_i^{\frac{1}{2}} \mathbf{P}_i \mathbf{T}_i^{\frac{1}{2}} \left(\sigma^2 \left[\mathbf{I}_N + c_i e_i^n\left(-\sigma^2\right) \mathbf{T}_i^{\frac{1}{2}} \mathbf{P}_i \mathbf{T}_i^{\frac{1}{2}}\right]\right)^{-1}$
$e_i^{n+1}\left(-\sigma^2\right) = \frac{1}{N} tr \mathbf{R}_i \left(\sigma^2 \left[\mathbf{I}_N + \sum_{k=1}^{K} \delta_k^n\left(-\sigma^2\right) \mathbf{R}_k\right]\right)^{-1}$
Update: $n = n + 1$ until $\max_j \left\{\left\|e_j^n - e_j^{n+1}\right\|\right\}$ is small enough.

MIMO-MAC capacity with precoders $\mathbf{P}_1,\ldots,\mathbf{P}_K$ before. Then we wish to determine the precoders for optimizing the capacity of MIMO-MAC. A proposition is provided to calculate precoding matrices $\left\{\mathbf{P}_1^0,\mathbf{P}_2^0,...,\mathbf{P}_K^0\right\}$ in [7].

Proposition: For $k\in\{1,\ldots,K\}$, denote $\mathbf{T}_k=\mathbf{U}_k\overline{\mathbf{T}}_k\mathbf{U}_k^H$ the spectral decomposition of $\mathbf{T}_k$ with $\mathbf{U}_k$ unitary and $\overline{\mathbf{T}}_k=diag\left(t_{k,1},\ldots,t_{k,n_k}\right)$. The eigenvectors of the precoders $\mathbf{P}_k^0=\mathbf{U}_k\overline{\mathbf{P}}_k^0\mathbf{U}_k^H$ coincide with the vectors of matrix $\mathbf{T}_k$ and the eigenvalue is given by $p_{k,i}^0=\left(\mu_k-1/c_ke_k^0t_{k,i}\right)^+$ which is the i^{th} diagonal entry of $\overline{\mathbf{P}}_k^0$. $(x)^+=\max\{0,x\}$, $e_k^0=e\left(-\sigma^2\right)$ and μ_k is chosen to satisfy the power constraint $1/n_k\, tr\mathbf{P}_k=P_k$.

Denote $\mathbf{U}_k=\begin{bmatrix}u_{11} & \cdots & u_{1n_k}\\ \vdots & \ddots & \vdots\\ u_{n_k1} & \cdots & u_{n_kn_k}\end{bmatrix}$, $\mathbf{P}_k=\begin{bmatrix}p_{k,1} & & \\ & \ddots & \\ & & p_{k,n_k}\end{bmatrix}$,

then $tr(\mathbf{P}_k)=\sum_{j=1}^{n_k}\sum_{i=1}^{n_k}u_{ji}u'_{ji}p_{k,i}=n_kP_k$ with $u'_{j,i}=conj\left(u_{j,i}\right)$.

In the following, we explain how to calculate $p_{k,i}$ to subject to the condition $1/n_k\, tr\mathbf{P}_k=P_k$ in Table II.

When there are no negative data in $p_{k,i}$, we get $p_{k,i}=\left(\mu_k-1/c_ke_k^0t_{k,i}\right)^+$ with power constraint. With the premise, it is easy to obtain optimal precoders from iterative water-filling results in Table III according to [7].

In Table III, we get the optimal precoding matrices for MIMO-MAC. In the following section, we compare uniform power allocation with optimal precoding policy depending on the deterministic equivalent for MIMO-MAC capacity.

TABLE II. POWER CONSTRAINT ALGORITHM

Initialization: $p_{k,i}=-1/c_ke_kt_{ki}, i\in\{1,\ldots,n_k\}$,

to satisfy power constraint,

$$\mu_k=\frac{n_kP_k-\sum_{j=1}^{n_k}\sum_{i=1}^{n_k}u_{ji}u'_{ji}p_{k,i}}{\sum_{j=1}^{n_k}\sum_{i=1}^{n_k}u_{ji}u'_{ji}},$$

then $p_{k,i}=p_{k,i}+\mu_k, i\in\{1,\ldots,k\}$

Iteration: while $\exists i\in\{1,\ldots,k\}, p_{k,i}<0$

take $p_{k,i}=0$ for those negative numbers, recount

$$\mu_k=\frac{n_kP_k-\sum_{j=1}^{n_k}\sum_{i=1}^{n_k}u_{ji}u'_{ji}p_{k,i}}{\sum_{j=1}^{n_k}\sum_{i=1}^{n_k}u_{ji}u'_{ji}},$$

$$p_{k,i}=p_{k,i}+\mu_k, i\in\{1,\ldots,k\}$$

TABLE III. ITERATIVE WATER-FILLING ALGORITHM

Initialization: Define the iteration step $n\geq 0$. When $n=0$, $p_{k,i}^0=P_k$ for all $k\in\{1,\ldots,K\}, i\in\{1,\ldots,n_k\}$.

Iteration n:

for $k\in\{1,\ldots,K\}$, $\mathbf{P}_k=\mathbf{U}_k\overline{\mathbf{P}}_k\mathbf{U}_k^H$ with $\overline{\mathbf{P}}_k=diag\left(p_{k,1}^n,\ldots,p_{k,n_k}^n\right)$,

then calculate e_k^{n+1}.

for $k\in\{1,\ldots,K\}$

$$p_{k,i}=\left(\mu_k-\frac{1}{c_ke_k^{n+1}t_{k,i}}\right)^+, i\in\{1,\ldots,n_k\}, \text{ based on Table II.}$$

end for

Update: $n=n+1$ until $\max_{k,i}\left\{\left|p_{k,i}^n-p_{k,i}^{n+1}\right|\right\}$ is small enough.

IV. SIMULATIONS AND RESULTS

In this section, computer simulations are conducted to evaluate the accuracy of the approximation C_{MAC}^0, and the effectiveness of the iterative algorithm developed in Table III. In particular, we are interested in their performances when the numbers of antennas are not so large. We assume a K-user MIMO-MAC channel with N antennas at base station and $n_1=n_2\ldots=n_K$ antennas at the user terminals, all placed in linear arrays. The space d_T between consecutive transmit antennas is such that $d_T/\lambda=1/2$, with λ the transmit signal wavelength; the distance d_R is the same for all users and such that $d_T=d_R$. The correlation matrices $\mathbf{T}_i, i\in\{1,\ldots,K\}$ at the users and $\mathbf{R}_j, j\in\{1,\ldots,N\}$ at the base station, are modelled according to [5] with Uniform, Gaussian and Laplacian power azimuth spectrum (PAS) having mean angle of ϕ_0 and root-mean-square of σ. Element in correlation matrices with Uniform PAS is

$$[\mathbf{T}/\mathbf{R}]_{m,n}=\int_{-\pi}^{\pi}\frac{1}{2\pi}\exp\left[j\frac{2\pi d_{T/R}(m-n)\sin(\phi)}{\lambda}\right]d\phi. \quad (7)$$

Element in correlation matrices with Gaussian PAS is

$$[\mathbf{T}/\mathbf{R}]_{m,n}=\int_{-\pi}^{\pi}\frac{Q}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(\phi-\phi_0)^2}{2\sigma^2}\right]\exp\left[j\frac{2\pi d_{T/R}(m-n)\sin(\phi)}{\lambda}\right]d\phi \quad (8)$$

with

$$Q=\frac{1}{erf\left(\frac{\pi}{\sigma\sqrt{2}}\right)}. \quad (9)$$

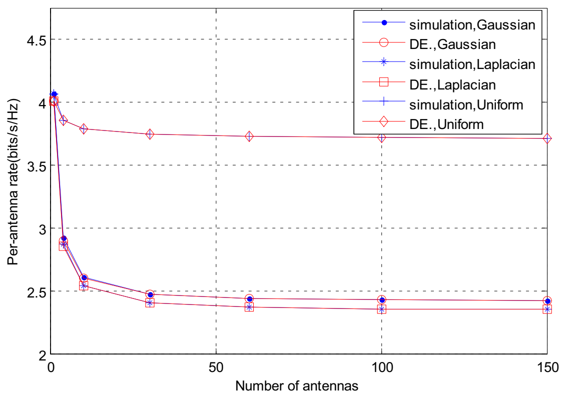

Figure 1. Ergodic per-antenna rates versus numbers of antennas with $N = n_1 = n_2$ and $SNR = 10$ dB. DE. represents deterministic equivalent.

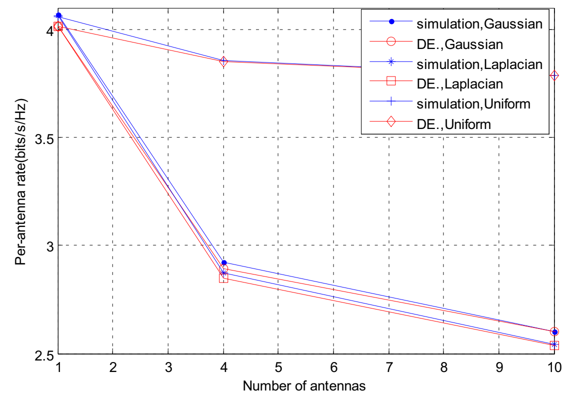

Figure 2. Detail of Fig.1 when antenna number is less than 10.

And element in correlation matrices with Laplacian PAS is

$$[\mathbf{T}/\mathbf{R}]_{m,n} = \int_{-\pi}^{\pi} \frac{Q}{\sigma\sqrt{2}} \exp\left[-\frac{\sqrt{2}|\phi - \phi_0|}{\sigma}\right] \exp\left[j\frac{2\pi d_{T/R}(m-n)\sin(\phi)}{\lambda}\right] d\phi \quad (10)$$

with

$$Q = \frac{1}{1 - \exp\left(-\frac{\sqrt{2}\pi}{\sigma}\right)}. \quad (11)$$

In Fig. 1, we consider a two-user MIMO-MAC with three kinds of PAS. We take the mean angels of the transmitters $\phi_{0,T_1} = 0, \phi_{0,T_2} = 30$ and the mean angels that arrive from the two users $\phi_{0,R_1} = 10, \phi_{0,R_2} = 40$. The root-mean-square value is the same for leave and arrival angels and such that $\sigma = 10$. With signal to noise ratio (SNR) of 10 dB, we take 10000 channel realizations to produce the simulated ergodic capacity under uniform power allocation and compared Monte-Carlo simulations against deterministic equivalent results with Uniform, Gaussian and Laplacian PAS in correlation matrices respectively. Fig. 2 is the detail of Fig. 1 when antenna number is less than 10. In the above experiments, we have shown that the deterministic equivalent provides a very good approximation for the per-antenna rate of finite dimensional systems with different types of fading distribution even when the numbers of antennas are not so large but the equation fits better as the growth of the number.

As we all know, the real channel conditions change all the time, whether the deterministic equivalent is suitable for different SNR environment is a question. In Fig. 3, we consider $N = n_1 = n_2 = 30$ and change SNR from -10 dB to 20 dB. We note that no matter good or bad channel status the deterministic equivalent results produce very good estimates, which makes our deterministic equivalent algorithm more generally. Moreover, the per-antenna rate increases faster with the growth of SNR. In Fig. 4, we simulate the optimal precoding policy i.e. iterative water-filling algorithm proposed in Table III with Gaussian PAS in correlation matrices, compared with uniform power allocation. It is obviously that the water-filling algorithm increases the per-antenna rate compared to the uniform power allocation. In addition, with the numbers of antennas increasing from 1 to 150, the per-antenna rate gap becomes wide first and when the antenna number comes to 30 the gap basically remain unchanged. Finally in order to prove the efficiency of the deterministic equivalent, we also compare the time used by the two means calculating MIMO-MAC capacity based on hypothesis of Fig. 1. In Fig. 5 we observe that when the numbers of antennas is less than 10, time interval between the two computing method can be neglected. But when the number is 60, the time interval significantly increases. The growth of the interval is even quicker than that of antenna number when each user has more than 100 antennas. From what has been discussed above, we may draw the conclusion that when the numbers of antennas grow the Monte-Carlo simulations will need rapid increasing time while the deterministic equivalent is much more efficient.

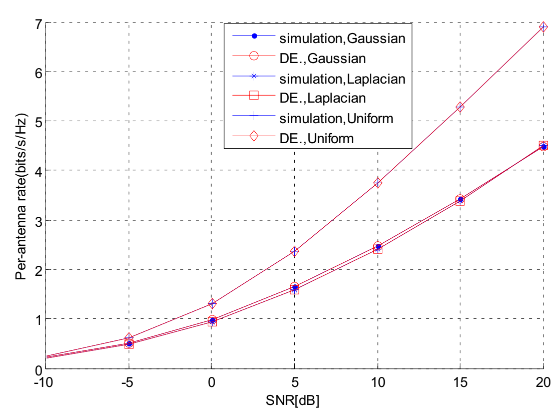

Figure 3. Ergodic per-antenna rates versus SNRs with $N = n_1 = n_2 = 30$.

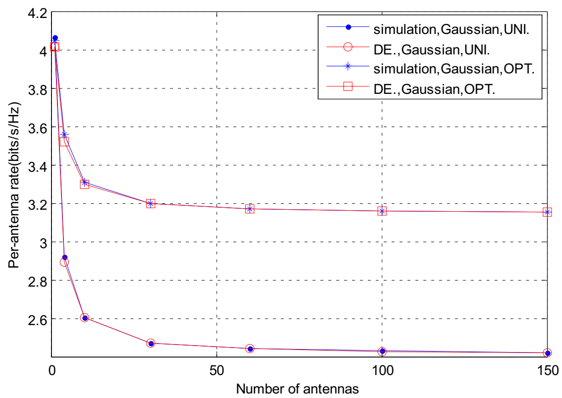

Figure 4. Ergodic per-antenna rates versus numbers of antennas with $N = n_1 = n_2$ and $SNR = 10$ dB. UNI. and OPT. represent uniform power allocation and optimal precoding respectively.

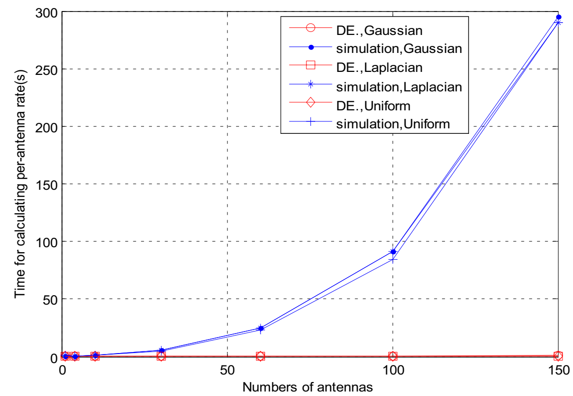

Figure 5. Times for calculating per-antenna rate versus numbers of antennas with $N = n_1 = n_2$ and $SNR = 10$ dB.

V. CONCLUSION

Based on RMT, we have developed specific algorithms for the performance analysis of multiuser uplink systems based on Kronecker channel model. We provide approximations of the ergodic capacity and precoding matrices which can be used to optimize system capacity. The simulations show perfect match with the theoretical formulas for different fading types even in the low-SNR region or with fewer antennas at the transmitters and receivers. In addition, the proposed performance approximations can be used to simulate the system behavior quickly without having to carry out extensive simulations.

REFERENCES

[1] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels," IEEE J. Sel. Areas Commun., vol. 51, no. 6, pp. 684–702, June 2003.

[2] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels," IEEE J. Sel. Areas Commun., vol. 51, no. 6, pp. 684–702, June 2003.

[3] J. H. Winters, "On the capacity of radio communication systems with diversity in a Rayleigh fading environment," IEEE J. Sel. Areas Commun., vol. SAC-5, pp. 871–878, June 1987.

[4] G. J. Foschini, " Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," Bell Labs Tech. J., vol. 1, no. 2, pp. 41–59, 1996.

[5] I. E. Telatar, "Capacity of multi-antenna gaussian channels," Eur. Trans. Telecom., vol. 10, pp. 585–595, November/December 1999.

[6] J. Fuhl, A. F. Molisch, and E. Bonek, "Unified channel model for mobile radio systems with smart antennas," Proc. Inst. Elect. Eng. Radar, Sonar Navig., vol. 145, no. 1, pp. 32–41, February 1998.

[7] M. Kiessling, "Unifying Analysis of Ergodic MIMO Capacity in Correlated Rayleigh Fading Environments," Eur. Trans. Telecom., vol. 16, no.1, pp. 17-35, January 2005.M. Young, The Technical Writer's Handbook. Mill Valley, CA: University Science, 1989.

[8] R. Couillet, M. Debbah, and J. W. Silverstein, "A deterministic equivalent for the capacity of correlated MIMO multiple access channels," IEEE Trans. Inf. Theory, vol. 57, no. 6, pp. 3493–3514, June 2011.

[9] J. Zhang, C. K. Wen, S. Jin, X. Gao, and K. K. Wong, "On capacity of large-scale MIMO multiple access channels with distributed sets of correlated antennas," IEEE J. Sel. Areas Commun., vol. 31, no. 2, pp. 133–148, February 2013.

[10] S. Wagner, R. Couillet, M. Debbah, and D. Slock, "Large system analysis of linear precoding in correlated MISO broadcast channels under limited feedback,' IEEE Trans. Inf. Theory, vol. 58, no. 7, pp. 4509–4537, July 2012.